

NORMAL FORM À LA MOSER FOR DIFFEOMORPHISMS AND GENERALIZATION OF RÜSSMANN'S TRANSLATED CURVE THEOREM TO HIGHER DIMENSION

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ABSTRACT. We prove a discrete time analogue of 1967 Moser's normal form of real analytic perturbations of vector fields possessing an invariant, reducible, Diophantine torus; in the case of diffeomorphisms too, the persistence of such an invariant torus is a phenomenon of finite co-dimension. Under convenient non-degeneracy assumptions on the diffeomorphisms under study (torsion property for example), this co-dimension can be reduced. As a by-product we obtain generalizations of Rüssmann's translated curve theorem in any dimension, by a technique of elimination of parameters.

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1. INTRODUCTION AND RESULTS

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $a, b \in \mathbb{R}$, $a < b$ and consider the twist map

$$P : \mathbb{T} \times [a, b] \rightarrow \mathbb{T} \times \mathbb{R}, \quad (\theta, r) \mapsto (\theta + \alpha(r), r),$$

where $\alpha'(r) > 0$: P preserves circles $r = r_0$, $r_0 \in [a, b]$, and *twist* them by an angle which increases as r does.

Moser in [21] proved that for any $r_0 \in (a, b)$ such that $\alpha(r_0)$ is Diophantine, if Q is an area preserving diffeomorphism sufficiently close to P , it has an invariant curve near $r = r_0$ on which the dynamics is conjugated to the rotation $\theta \mapsto \theta + \alpha(r_0)$.

In 1970, Rüssmann generalized this fundamental result to non-conservative twist diffeomorphisms of the annulus [3, 24, 29]. He showed that the persistence of a Diophantine, invariant circle is a phenomenon of co-dimension 1: in general the invariant curve does not persist but is translated in the normal direction. It is the

”theorem of the translated curve” (see below for a precise statement).

As in Kolmogorov’s theorem [17], the dynamics on the translated curve can be conjugated to the same initial Diophantine rotation because of the non degeneracy (twist) of the map. Herman gave a proof of the translated curve theorem for diffeomorphisms with rotation number of constant type [15], then generalized Rüssmann’s result in higher dimension to diffeomorphisms of $\mathbb{T}^n \times \mathbb{R}$ ($\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$) close enough to the rotation $(\theta, r) \mapsto (\theta + 2\pi\alpha, r)$, $2\pi\alpha$ being a Diophantine vector, without assuming any twist hypothesis but introducing an external parameter in order to tune the frequency on the translated torus, yet breaking the dynamical conjugacy to the Diophantine rotation, see [29].

Up to our knowledge no further generalization in $\mathbb{T}^n \times \mathbb{R}^n$ of Rüssmann’s theorem has been given so far.

The first purpose of this work is to prove a discrete-time analogue of Moser’s 1967 normal form [23] of real analytic perturbations of vector fields on $\mathbb{T}^n \times \mathbb{R}^m$ possessing a quasi-periodic Diophantine, reducible, invariant torus. The normal form will then be used to deduce a ”translated torus theorem” under convenient non-degeneracy assumptions. As a by-product, Rüssmann’s classic theorem will be a particular case of small dimension. While Rüssmann and Herman consider smooth or finite differentiable diffeomorphisms, we focus here on the analytic category. Let us state the main results.

A normal form for diffeomorphisms. Let $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ be the n -dimensional torus. Let V be the space of germs along $\mathbb{T}^n \times \{0\}$ in $\mathbb{T}^n \times \mathbb{R}^m = \{(\theta, r)\}$ of real analytic diffeomorphisms. Fix $\alpha \in \mathbb{R}^n$ and $A \in \text{Mat}_m(\mathbb{R})$, assuming that A is diagonalizable with eigenvalues $a_1, \dots, a_m \in \mathbb{C}$ different from 0.

Let $U(\alpha, A)$ be the affine subspace of V of diffeomorphisms of the form

$$(1.1) \quad P(\theta, r) = (\theta + 2\pi\alpha + O(r), A \cdot r + O(r^2)),$$

where $O(r^k)$ are terms of order $\geq k$ in r which may depend on θ . For these diffeomorphisms $\mathbb{T}_0^n = \mathbb{T}^n \times \{0\}$ is an invariant, reducible, α -quasi-periodic torus whose normal dynamics at the first order is characterized by a_1, \dots, a_m . We will collectively refer to $\alpha_1, \dots, \alpha_n$ and a_1, \dots, a_m as the *characteristic frequencies* or *characteristic numbers* of \mathbb{T}_0^n .

Let $\arg a = (\arg a_1, \dots, \arg a_r) \in \mathbb{R}^r$ ($0 \leq r < m$) be the vector of the arguments of those a_i ’s having positive imaginary part, say $a_i = \rho_i e^{i \arg a_i}$, where $\rho > 0$, $0 < \arg a_i < \pi$, and assume that the characteristic numbers satisfy the following Diophantine condition for some real $\gamma, \tau > 0$

$$(1.2) \quad |2\pi k \cdot \alpha - h \cdot \arg a - 2\pi l| \geq \frac{\gamma}{(1 + |k|)^\tau}, \quad \forall k \in \mathbb{N}^n \setminus \{0\}, \forall (l, h) \in \mathbb{Z} \times \mathbb{Z}^r, |h| \leq 2.$$

Let \mathcal{G} be the space of germs of real analytic isomorphisms of $\mathbb{T}^n \times \mathbb{R}^m$ of the form

$$(1.3) \quad G(\theta, r) = (\varphi(\theta), R_0(\theta) + R_1(\theta) \cdot r),$$

where φ is a diffeomorphism of the torus fixing the origin and R_0, R_1 are functions defined on the torus \mathbb{T}^n with values in \mathbb{R}^n and $\text{Mat}_m(\mathbb{R})$ respectively.

Eventually, let us define the "translation map"

$$T_\lambda : \mathbb{T}^n \times \mathbb{R}^m \rightarrow \mathbb{T}^n \times \mathbb{R}^m, \quad (\theta, r) \mapsto (\beta + \theta, b + (I + B) \cdot r),$$

where $\beta \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $B \in \text{Mat}_m(\mathbb{R})$ are such that

$$(1.4) \quad (A - I) \cdot b = 0, \quad [A, B] = 0,$$

having denoted with I the identity matrix in $\text{Mat}_m(\mathbb{R})$.

We will refer to translating parameters $\lambda = (\beta, b + B \cdot r)$ as *corrections* or *counter terms*, and denote with Λ the space of such λ 's

$$\Lambda = \{\lambda = (\beta, b + B \cdot r) : (A - I) \cdot b = 0, [A, B] = 0\}.$$

Theorem A (Normal form). *If Q is sufficiently close to $P^0 \in U(\alpha, A)$, there exists a unique triplet $(G, P, \lambda) \in \mathcal{G} \times U(\alpha, A) \times \Lambda$, close to $(\text{id}, P^0, 0)$, such that*

$$Q = T_\lambda \circ G \circ P \circ G^{-1}.$$

In the neighborhood of $(\text{id}, P^0, 0)$, the \mathcal{G} -orbit of all P 's $\in U(\alpha, A)$ has finite co-dimension. The proof is based on a relatively general inverse function theorem in analytic class (Theorem A.1 of the Appendix).

The idea of proving the finite co-dimension of a set of conjugacy classes of a diffeomorphism or of a vector field has been successfully exploited by many authors. Arnold at first proved a normal form for diffeomorphisms of \mathbb{T}^n [1], followed by Moser's normal forms for vector fields [18, 22, 23, 28]. Among other authors we recall Calleja-Celletti-deLaLlave work on conformally symplectic systems [4], Chenciner's study on the bifurcation of elliptic fixed points [5–7], Herman's twisted conjugacy for Hamiltonians [11, 12] (a generalization of Arnold's work [1]) or Eliasson-Fayad-Krikorian work around the stability of KAM tori [9].

This technique allows us to study the persistence of an invariant torus in two steps: first, prove a normal form that does not depend on any non-degeneracy hypothesis (but that contains the hard analysis); second, reduce or eliminate the (finite dimensional) corrections by the usual implicit function theorem, using convenient non degeneracy assumptions on the system under study.

A generalization of Rüssmann's theorem. From the normal form of Theorem A, we see that when $\lambda = 0$, $Q = G \circ P \circ G^{-1}$: the torus $G(\mathbb{T}_0^n)$ is invariant for Q and the first order dynamics is given by $P \in U(\alpha, A)$. Conversely, whenever $\lambda = (\beta, b)$, the torus is *translated* and the $2\pi\alpha$ -quasi-periodic tangential dynamics is *twisted* by the correction in β :

$$Q(\varphi(\theta), R_0(\theta)) = (\beta + \varphi(\theta + 2\pi\alpha), b + R_0(\theta + 2\pi\alpha)).$$

We will loosely say that the torus \mathbb{T}_0^n

- persists up to twist-translation, when $\lambda = (\beta, b)$
- persists up to translation, when $\lambda = (0, b)$

We stress the fact that Theorem A not only gives the tangential dynamics to the torus, but also the normal one, of which Rüssmann's original statement is regardless:

Theorem (Rüssmann). *Let $\alpha \in \mathbb{R}$ be Diophantine and $P^0 : \mathbb{T} \times [-r_0, r_0] \rightarrow \mathbb{T} \times \mathbb{R}$ be of the form*

$$P^0(\theta, r) = (\theta + 2\pi\alpha + t(r) + O(r^2), (1 + A^0)r + O(r^2)),$$

where $A^0 \in \mathbb{R} \setminus \{-1\}$, $t(0) = 0$ and $t'(r) > 0$.

If Q is close enough to P^0 there exists a unique analytic curve $\gamma : \mathbb{T} \rightarrow \mathbb{R}$, close to $r = 0$, an analytic diffeomorphism φ of \mathbb{T} close to the identity and $b \in \mathbb{R}$, close to 0, such that

$$Q(\theta, \gamma(\theta)) = (\varphi \circ R_{2\pi\alpha} \circ \varphi^{-1}(\theta), b + \gamma(\varphi \circ R_{2\pi\alpha} \circ \varphi^{-1}(\theta))).$$

In the original statement $A^0 = 0$; to consider this case does not add any difficulty to the proof.

We will generalize Rüssmann's theorem on $\mathbb{T}^n \times \mathbb{R}^n$. At the expense of conjugating $T_\lambda^{-1} \circ Q$ to a diffeomorphism P whose invariant torus has different constant normal dynamics A , under convenient non-degeneracy conditions we can prove the existence of a twisted-translated or translated α -quasi-periodic Diophantine torus by application of the classic implicit function theorem in finite dimension. The following results will be proved in section 5, where a more functional statement will be given (Theorem 5.1 and 5.2).

On $\mathbb{T}^n \times \mathbb{R}^n$, let $P \in U(\alpha, A)$, defined in expression (1.1), be such that A has simple, real, non 0 eigenvalues a_1, \dots, a_n . This hypothesis clearly implies that the only frequencies that can cause small divisors are the tangential ones $\alpha_1, \dots, \alpha_n$, so that we only need to require the standard Diophantine hypothesis on α .

Theorem B. *If Q is sufficiently close to $P^0 \in U(\alpha, A)$, the torus T_0^n persists up to twist-translation.*

If, in addition, Q has a torsion property we can prove the following theorem.

Theorem C. *Let*

$$P^0(\theta, r) = (\theta + 2\pi\alpha + p_1(\theta) \cdot r + O(r^2), (I + A^0) \cdot r + O(r^2)),$$

be such that

$$\det \left(\int_{\mathbb{T}^n} p_1(\theta) d\theta \right) \neq 0.$$

If Q is sufficiently close to P^0 , the torus T_0^n persists up to translation.

The paper is organized as follows: in sections 2-3 we introduce the normal form operator, define conjugacy spaces and present the difference equations that will be solved to linearize the dynamics on the perturbed torus; in section 4 we will prove Theorem A while in section 5 we will prove Theorems B and C.

2. THE NORMAL FORM OPERATOR

We will show that the operator

$$\phi : \mathcal{G} \times U(\alpha, A) \times \Lambda \rightarrow V, \quad (G, P, \lambda) \mapsto T_\lambda \circ G \circ P \circ G^{-1}$$

is a local diffeomorphism (in the sense of scales of Banach spaces) in a neighborhood of $(\text{id}, P^0, 0)$. Note that ϕ is formally defined on the whole space but $\phi(G, P, \lambda)$ is analytic in the neighborhood of T_0^n only if G is close enough to the identity with respect to the width of analyticity of P . See section 2.3.

Although the difficulty to overcome in the proof is rather standard for conjugacy problems of this kind (proving the fast convergence of a Newton-like scheme), the procedure relies on a relatively general inverse function theorem (Theorem A.1 of section A), following a strategy alternative to Zehnder's in [30]. Both Zehnder's approach and ours rely on the fact that the fast convergence of the Newton's scheme is somewhat independent of the internal structure of the variables.

2.1. Complex extensions. Let us extend the tori

$$\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n \quad \text{and} \quad \mathbb{T}_0^n = \mathbb{T}^n \times \{0\} \subset \mathbb{T}^n \times \mathbb{R}^m,$$

as

$$\mathbb{T}_{\mathbb{C}}^n = \mathbb{C}^n / 2\pi\mathbb{Z}^n \quad \text{and} \quad \mathbb{T}_{\mathbb{C}}^n = \mathbb{T}_{\mathbb{C}}^n \times \mathbb{C}^m$$

respectively, and consider the corresponding s -neighborhoods defined using ℓ^∞ -balls (in the real normal bundle of the torus):

$$\mathbb{T}_s^n = \left\{ \theta \in \mathbb{T}_{\mathbb{C}}^n : \max_{1 \leq j \leq n} |\text{Im } \theta_j| \leq s \right\} \quad \text{and} \quad \mathbb{T}_s^n = \{(\theta, r) \in \mathbb{T}_{\mathbb{C}}^n : |(\text{Im } \theta, r)| \leq s\},$$

where $|(\text{Im } \theta, r)| := \max_{1 \leq j \leq n} \max(|\text{Im } \theta_j|, |r_j|)$.

Let now $f : \mathbb{T}_s^n \rightarrow \mathbb{C}$ be real holomorphic on the interior of \mathbb{T}_s^n , continuous on \mathbb{T}_s^n , and consider its Fourier expansion $f(\theta, r) = \sum_{k \in \mathbb{Z}^n} f_k(r) e^{i k \cdot \theta}$, noting $k \cdot \theta = k_1 \theta_1 + \dots + k_n \theta_n$. In this context we introduce the so called "weighted norm":

$$|f|_s := \sum_{k \in \mathbb{Z}^n} |f_k| e^{|k|s}, \quad |k| = |k_1| + \dots + |k_n|,$$

$|f_k| = \sup_{|r| < s} |f_k(r)|$. Whenever $f : \mathbb{T}_s^n \rightarrow \mathbb{C}$, $|f|_s = \max_{1 \leq j \leq n} (|f_j|_s)$, f_j being the j -th component of $f(\theta, r)$.

It is a trivial fact that the classical sup-norm is bounded from above by the weighted norm:

$$\sup_{z \in \mathbb{T}_s^n} |f(z)| \leq |f|_s$$

and that $|f|_s < +\infty$ whenever f is analytic on its domain, which necessarily contains some $\mathbb{T}_{s'}^n$ with $s' > s$.¹ In addition, the following useful inequalities hold if f, g are analytic on $\mathbb{T}_{s'}^n$

$$|f|_s \leq |f|_{s'} \quad \text{for } 0 < s < s',$$

¹The inequality shows the well known fact that if f is real analytic on \mathbb{T}^n , it admits a holomorphic bounded extension: its Fourier's coefficients decay esponentially and there exists $s > 0$ such that $|f|_s < \infty$

and

$$|fg|_{s'} \leq |f|_{s'} |g|_{s'}.$$

For more details about the weighted norm, see for example [20].

In general for complex extensions U_s and $V_{s'}$, we will denote by $\mathcal{A}(U_s, V_{s'})$ the set of holomorphic functions from U_s to $V_{s'}$ and $\mathcal{A}(U_s)$, endowed with the s -weighted norm, the Banach space $\mathcal{A}(U_s, \mathbb{C})$.

Eventually, let E and F be two Banach spaces,

- We indicate contractions with a dot “ \cdot ”, with the convention that if $l_1, \dots, l_{k+p} \in E^*$ and $x_1, \dots, x_p \in E$

$$(l_1 \otimes \dots \otimes l_{k+p}) \cdot (x_1 \otimes \dots \otimes x_p) = l_1 \otimes \dots \otimes l_k \langle l_{k+1}, x_1 \rangle \dots \langle l_{k+p}, x_p \rangle.$$

In particular, if $l \in E^*$, we simply write $l^n = l \otimes \dots \otimes l$.

- If f is a differentiable map between two open sets of E and F , $f'(x)$ is considered as a linear map belonging to $F \otimes E^*$, $f'(x) : \zeta \mapsto f'(x) \cdot \zeta$; the corresponding norm will be the standard operator norm

$$|f'(x)| = \sup_{\zeta \in E, |\zeta|_E=1} |f'(x) \cdot \zeta|_F.$$

2.2. Spaces of conjugacies.

- We consider the set \mathcal{G}_s^σ of germs of holomorphic diffeomorphisms on T_s^n such that

$$|\varphi - \text{id}|_s \leq \sigma$$

and

$$|R_0 + (R_1 - \text{id}) \cdot r|_s \leq \sigma,$$

and endow the tangent space at the identity $T_{\text{id}} \mathcal{G}_s^\sigma$ with the norm

$$|\dot{G}|_s = \max_{1 \leq j \leq n+m} \left(|\dot{G}_j|_s \right).$$

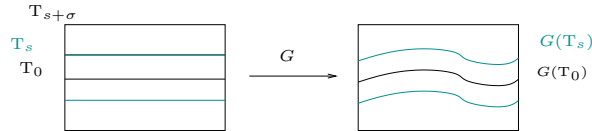


FIGURE 1. Deformed complex domain

- Let V_s be the subspace of $\mathcal{A}(T_s^n, \mathbb{T}_{\mathbb{C}}^n \times \mathbb{C}^m)$ of diffeomorphisms

$$Q : (\theta, r) \mapsto (f(\theta, r), g(\theta, r)),$$

where $f \in \mathcal{A}(T_s^n, \mathbb{C}^n)$, $g \in \mathcal{A}(T_s^n, \mathbb{C}^m)$, endowed with the norm

$$|Q|_s = \max(|f|_s, |g|_s).$$

- Let $U_s(\alpha, A)$ be the subspace of V_s of those diffeomorphisms P of the form

$$P(\theta, r) = (\theta + 2\pi\alpha + O(r), A \cdot r + O(r^2)).$$

We will indicate with p_i and P_i the coefficients of the order- i term in r , in the θ and r -directions respectively.

- If $G \in \mathcal{G}_s^\sigma$ and P is a diffeomorphism over $G(\mathbb{T}_s^n)$ we define the following deformed norm

$$|P|_{G,s} := |P \circ G|_s,$$

depending on G ; this in order not to shrink artificially the domains of analyticity. The problem, in a smooth context, may be solved without changing the domain, by using plateau functions.

2.3. The normal form operator. By Theorem B.1 and Corollary B.1 the following operator

$$(2.1) \quad \begin{array}{ccc} \phi : & \mathcal{G}_{s+\sigma}^{\sigma/n} \times U_{s+\sigma}(\alpha, A) \times \Lambda & \rightarrow V_s \\ & (G, P, \lambda) & \mapsto T_\lambda \circ G \circ P \circ G^{-1} \end{array}$$

is now well defined. It would be more appropriate to write $\phi_{s,\sigma}$ but, since these operators commute with source and target spaces, we will refer to them simply as ϕ . We will always assume that $0 < s < s + \sigma < 1$ and $\sigma < s$.

3. DIFFERENCE EQUATIONS

We will apply the following Lemmata to linearize the tangent and the normal dynamics of the torus (see section 4).

Let $\alpha \in \mathbb{R}^n$ and $\arg a = (\arg a_1, \dots, \arg a_r) \in \mathbb{R}^r$ ($0 \leq r < m$), the vector of arguments of complex eigenvalues of $A \in \text{Mat}_m(\mathbb{R})$ with positive imaginary part, satisfy the following conditions, which all follow from (1.2).

$$(3.1) \quad |k \cdot \alpha - l| \geq \frac{\gamma}{|k|^\tau}, \quad \forall k \in \mathbb{N}^n \setminus \{0\}, \forall l \in \mathbb{Z}$$

$$(3.2) \quad |2\pi k \cdot \alpha - \arg a_j - 2\pi l| \geq \frac{\gamma}{(1 + |k|)^\tau}, \quad \forall k \in \mathbb{N}^n, \forall l \in \mathbb{Z}, j = 1, \dots, r,$$

$$(3.3) \quad |2\pi k \cdot \alpha + h \cdot \arg a - 2\pi l| \geq \frac{\gamma}{(1 + |k|)^\tau}, \quad \forall (k, h) \in \mathbb{N}^n \times \mathbb{Z}^r \setminus \{0\}, \forall l \in \mathbb{Z}, \quad |h| = 2.$$

The following fundamental Lemma is the heart of the proof of Theorem A and, more generally, of many stability results related to Diophantine rotations on the torus.

Lemma 1 (Straightening the tangent dynamics). *Let $\alpha \in \mathbb{R}$ be Diophantine in the sense of (3.1). For any $g \in \mathcal{A}(\mathbb{T}_{s+\sigma})$, there exists a unique $f \in \mathcal{A}(\mathbb{T}_s)$ of zero average and a unique $\mu \in \mathbb{R}$ such that*

$$\mu + f(\theta + 2\pi\alpha) - f(\theta) = g(\theta), \quad \mu = \int_{\mathbb{T}} g,$$

satisfying

$$|f|_s \leq \frac{C}{\gamma \sigma^{\tau+1}} |g|_{s+\sigma},$$

C being a constant depending only on τ .

Complement. For any $a, b \in \mathbb{R}^+ \setminus \{0\}$, $a \neq b$, and any $g \in \mathcal{A}(\mathbb{T}_{s+\sigma})$ there exists a unique $f \in \mathcal{A}(\mathbb{T}_s)$ such that

$$af(\theta + 2\pi\alpha) - bf(\theta) = g(\theta),$$

satisfying the same kind of estimate.

Proof. Developing in Fourier series yields

$$\sum_k (e^{i2\pi k\alpha} - 1) f_k e^{ik\theta} = \sum_k g_k e^{ik\theta},$$

letting $\mu = g_0$ we formally have

$$f(\theta) = \sum_{k \neq 0} \frac{g_k}{e^{i2\pi k\alpha} - 1} e^{ik\theta}.$$

To prove the estimate, remark that for any $a, b \in \mathbb{R}^+$

$$\begin{aligned} |a e^{i2\pi k\alpha} - b|^2 &= (a - b)^2 \cos^2 \frac{2\pi k\alpha}{2} + (a + b)^2 \sin^2 \frac{2\pi k\alpha}{2} \\ &\geq (a + b)^2 \sin^2 \frac{2\pi k\alpha}{2} = (a + b)^2 \sin^2 \frac{2\pi(k\alpha - l)}{2}, \end{aligned}$$

with $l \in \mathbb{Z}$. Choosing $l \in \mathbb{Z}$ such that $-\frac{\pi}{2} \leq \frac{2\pi(k\alpha - l)}{2} \leq \frac{\pi}{2}$, we get

$$|e^{i2\pi k\alpha} - 1| \geq 8\pi^{-2} |k\alpha - l| \geq 8\pi^{-2} \frac{\gamma}{|k|^\tau},$$

by the classical inequality $|\sin x| \geq \frac{2}{\pi}|x|$, whenever $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, and condition (3.1). To get the claimed estimate is now a standard computation. We address the reader interested to optimal estimates (with σ^τ instead of $\sigma^{\tau+1}$) to [25]. The proof of the complement is straightforward. \square

Remark 3.1. Note that the homological equation of the complement does not involve small divisors and it can readily be solved, without losing analyticity, just bounding the denominator from above with $|a - b|$. Small divisors can occur only in the case $a = b$ or $|a| = |b| = 1$.

Let now $\alpha \in \mathbb{R}^n$ and $A \in \text{Mat}_m(\mathbb{R})$ be such that $a_i \neq 1, i = 1, \dots, m$, and consider the following operator

$$L_{1,A} : \mathcal{A}(\mathbb{T}_{s+\sigma}^n, \mathbb{C}^m) \rightarrow \mathcal{A}(\mathbb{T}_s^n, \mathbb{C}^m), \quad f \mapsto f(\theta + 2\pi\alpha) - A \cdot f(\theta).$$

Lemma 2 (Relocating the torus). *Let $\alpha \in \mathbb{R}^n$ and $A \in \text{Mat}_m(\mathbb{R})$ be a diagonalizable matrix, with eigenvalues distinct from 1, satisfying the Diophantine condition (3.2). For every $g \in \mathcal{A}(\mathbb{T}_{s+\sigma}^n, \mathbb{C}^m)$, there exists a unique preimage $f \in \mathcal{A}(\mathbb{T}_s^n, \mathbb{C}^m)$ by $L_{1,A}$. Moreover the following estimate holds*

$$|f|_s \leq \frac{C_2}{\gamma} \frac{1}{\sigma^{n+\tau}} |g|_{s+\sigma},$$

C_2 being a constant depending only on the dimension n and the exponent τ .

Proof. Let us first suppose that A is diagonal.

Expanding both sides of $L_{1,A}f = g$ we see that the Fourier coefficient of the j -th component of f is given by

$$f_k^j = \frac{g_k^j}{e^{i2\pi k\alpha} - a_j},$$

and the proof is straightforward from Lemma 1, once we write those negative a_j as $a_j = |a_j|e^{i\pi}$.

When A is diagonalizable, let $P \in \text{GL}_m(\mathbb{C})$ be the transition matrix such that PAP^{-1} is diagonal. Considering $f(\theta + 2\pi\alpha) - A \cdot f(\theta)$, and left multiplying both sides by P , we get

$$\tilde{f}(\theta + 2\pi\alpha) + PAP^{-1}\tilde{f}(\theta) = \tilde{g},$$

where we have set $\tilde{g} = Pg$ and $\tilde{f} = Pf$. This equation has a unique solution with the wanted estimates following Lemma 1, once we take care for those complex $a_j = \rho_j e^{i \arg a_j}$, $j = 1, \dots, r$, and write the denominator as

$$e^{i2\pi k\alpha} - a_j = e^{i \arg a_j} \left(e^{i(2\pi k\alpha - \arg a_j)} - \rho_j \right).$$

Taking into account Diophantine conditions (3.1) and (3.2), we get the wanted thesis. We just need to put $f = P^{-1}\tilde{f}$. \square

Eventually, consider a holomorphic function F on $\mathbb{T}_{s+\sigma}^n$ with values in $\text{Mat}_m(\mathbb{C})$ and define the operator

$$\begin{array}{ccc} L_{2,A} : \mathcal{A}(\mathbb{T}_{s+\sigma}^n, \text{Mat}_m(\mathbb{C})) & \rightarrow & \mathcal{A}(\mathbb{T}_s^n, \text{Mat}_m(\mathbb{C})) \\ F & \mapsto & F(\theta + 2\pi\alpha) \cdot A - A \cdot F(\theta) \end{array}.$$

Lemma 3 (Straighten the first order dynamics). *Let $\alpha \in \mathbb{R}^n$ and $A \in \text{Mat}_m(\mathbb{R})$ be a diagonalizable matrix, with eigenvalues distinct from 1, satisfying the Diophantine conditions (3.1) and (3.3) respectively. For every $G \in \mathcal{A}(\mathbb{T}_{s+\sigma}^n, \text{Mat}_m(\mathbb{C}))$, such that $\int_{\mathbb{T}^n} G_i^i \frac{d\theta}{(2\pi)^n} = 0$, there exists a unique $F \in \mathcal{A}(\mathbb{T}_s^n, \text{Mat}_m(\mathbb{C}))$, having zero average diagonal elements, such that the matrix equation*

$$F(\theta + 2\pi\alpha) \cdot A - A \cdot F(\theta) = G(\theta)$$

is satisfied; moreover the following estimate holds

$$|F|_s \leq \frac{C_3}{\gamma} \frac{1}{\sigma^{n+\tau}} |G|_{s+\sigma},$$

C_3 being a constant depending only on the dimension n and the exponent τ .

Proof. Let $A = \text{diag}(a_1, \dots, a_m) \in \mathbb{R}^m$ be diagonal and $F \in \text{Mat}_m(\mathbb{C})$ be given, expanding $L_{2,A}F = G$ we get m equations of the form

$$a_j \left(F_j^j(\theta + 2\pi\alpha) - F_j^j(\theta) \right) = G_j^j, \quad j = 1, \dots, m$$

and $m^2 - m$ equations of the form

$$a_j F_j^i(\theta + 2\pi\alpha) - a_i F_j^i(\theta) = G_j^i(\theta), \quad \forall i \neq j, i, j = 1, \dots, m.$$

where we denoted F_j^i the element corresponding to the i -th line and j -th column of the matrix $F(\theta)$. Taking into account condition (3.1), the thesis follows from the

same computations of Lemma 2.

Eventually, to recover the general case, we consider the transition matrix $P \in \text{GL}_m(\mathbb{C})$ and the equation

$$(PF(\theta + 2\pi\alpha)P^{-1}PAP^{-1}) - PAP^{-1}PF(\theta)P^{-1} = PGP^{-1};$$

letting $\tilde{F} = PFP^{-1}$ and $\tilde{G} = PGP^{-1}$, the equation is of the previous kind and by conditions (3.1)-(3.3), via the same kind of calculations, we get the thesis. It remains to recover $G = P^{-1}\tilde{G}P$. \square

4. INVERSION OF THE OPERATOR ϕ

The following theorem represents the main result of this first part, from which the normal form Theorem A follows.

Let us fix $P^0 \in U_s(\alpha, A)$ and note $V_s^\sigma = \{Q \in V : |Q - P^0|_s < \sigma\}$ the ball of radius σ centered at P^0 .

Theorem 4.1. *The operator ϕ is a local diffeomorphism in the sense that for every $s < s + \sigma < 1$ there exists $\varepsilon > 0$ and a unique C^∞ -map ψ*

$$\psi : V_{s+\sigma}^\varepsilon \rightarrow \mathcal{G}_s \times U_s(\alpha, A) \times \Lambda$$

such that $\phi \circ \psi = \text{id}$. Moreover ψ is Whitney-smooth with respect to (α, A) .

This result will follow from the inverse function theorem A.1 and regularity propositions A.2-A.1-A.3.

In order to solve locally $\phi(x) = y$, we use the remarkable idea of Kolmogorov and find the solution by composing infinitely many times the operator

$$x = (g, u, \lambda) \mapsto x + \phi'^{-1}(x) \cdot (y - \phi(x)),$$

on extensions $T_{s+\sigma}^n$ of shrinking width.

At each step of the induction, it is necessary that $\phi'^{-1}(x)$ exists at an unknown x (not only at x_0) in a whole neighborhood of x_0 and that ϕ'^{-1} and ϕ'' satisfy a suitable estimate, in order to control the convergence of the iterates.

The main step is to check the existence of a right inverse for

$$\phi'(G, P, \lambda) : T_G \mathcal{G}_{s+\sigma}^{\sigma/n} \times \vec{U}_{s+\sigma} \times \Lambda \rightarrow V_{G,s},$$

if G is close to the identity. We indicated with \vec{U} the vector space directing $U(\alpha, A)$.

Proposition 4.1. *There exists ε_0 such that if $(G, P, \lambda) \in \mathcal{G}_{s+\sigma}^{\varepsilon_0} \times U_{s+\sigma}(\alpha, A) \times \Lambda$, for all $\delta Q \in V_{G,s+\sigma} = G^* \mathcal{A}(T_{s+\sigma}^n, \mathbb{C}^{n+m})$, there exists a unique triplet $(\delta G, \delta P, \delta \lambda) \in T_G \mathcal{G}_s \times \vec{U}_s \times \Lambda$ such that*

$$(4.1) \quad \phi'(G, P, \lambda) \cdot (\delta G, \delta P, \delta \lambda) = \delta Q.$$

Moreover we have the following estimates

$$(4.2) \quad \max(|\delta G|_s, |\delta P|_s, |\delta \lambda|) \leq \frac{C'}{\sigma^{\tau'}} |\delta Q|_{G,s+\sigma},$$

C' being a constant possibly depending on $|((G - \text{id}), P - (\theta + 2\pi\alpha, A \cdot r))|_{s+\sigma}$.

Proof. Differentiating with respect to $x = (G, P, \lambda)$, we have

$$\delta(T_\lambda \circ G \circ P \circ G^{-1}) = T_{\delta\lambda} \circ (G \circ P \circ G^{-1}) + T'_\lambda \circ (G \circ P \circ G^{-1}) \cdot \delta(G \circ P \circ G^{-1})$$

hence

$$M \cdot (\delta G \circ P + G' \circ P \cdot \delta P - G' \circ P \cdot P' \cdot G'^{-1} \cdot \delta G) \circ G^{-1} = \delta Q - T_{\delta\lambda} \circ (G \circ P \circ G^{-1}),$$

$$\text{where } M = \begin{pmatrix} I & 0 \\ 0 & I + B \end{pmatrix}.$$

The data is δQ while the unknowns are the "tangent vectors" $\delta P \in O(r) \times O(r^2)$, δG (geometrically, a vector field along G) and $\delta\lambda \in \Lambda$.

Pre-composing by G we get the equivalent equation between germs along the standard torus T_0^n (as opposed to $G(T_0^n)$):

$$M \cdot (\delta G \circ P + G' \circ P \cdot \delta P - G' \circ P \cdot P' \cdot G'^{-1} \cdot \delta G) = \delta Q \circ G - T_{\delta\lambda} \circ G \circ P;$$

multiplying both sides by $(G'^{-1} \circ P)M^{-1}$, we finally obtain

$$(4.3) \quad \dot{G} \circ P - P' \cdot \dot{G} + \delta P = G'^{-1} \circ P \cdot M^{-1} \delta Q \circ G + G'^{-1} \circ P \cdot M^{-1} T_{\delta\lambda} \circ G \circ P,$$

where $\dot{G} = G'^{-1} \cdot \delta G$.

Remark that the term containing $T_{\delta\lambda}$ is not constant; expanding along $r = 0$, it reads

$$T_\lambda = G'^{-1} \circ P \cdot M^{-1} \cdot T_{\delta\lambda} \circ G \circ P = (\dot{\beta} + O(r), \dot{b} + \dot{B} \cdot r + O(r^2)).$$

The vector field \dot{G} (geometrically, a germ along T_0^n of tangent vector fields) reads

$$\dot{G}(\theta, r) = (\dot{\varphi}(\theta), \dot{R}_0(\theta) + \dot{R}_1(\theta) \cdot r).$$

The problem is now: G, λ, P, Q being given, find $\dot{G}, \delta P$ and $\dot{\lambda}$, hence $\delta\lambda$ and δG .

We are interested in solving the equation up to the 0-order in r in the θ -direction, and up to the first order in r in the action direction; hence we consider the Taylor expansions along T_0^n up to the needed order.

We remark that since $\delta P = (O(r), O(r^2))$, it will not intervene in the cohomological equations given out by (4.3), but will be uniquely determined by identification of the reminders.

Let us proceed to solve the equation (4.3), which splits into the following three

$$(4.4) \quad \begin{aligned} \dot{\varphi}(\theta + 2\pi\alpha) - \dot{\varphi}(\theta) + p_1 \cdot \dot{R}_0 &= \dot{q}_0 + \dot{\beta} \\ \dot{R}_0(\theta + 2\pi\alpha) - A \cdot \dot{R}_0(\theta) &= \dot{Q}_0 + \dot{b} \\ \dot{R}_1(\theta + 2\pi\alpha) \cdot A - A \cdot \dot{R}_1(\theta) &= \dot{Q}_1 - (2P_2 \cdot \dot{R}_0 + \dot{R}_0'(\theta + 2\pi\alpha) \cdot p_1) + \dot{B}. \end{aligned}$$

The first equation is the one straightening the tangential dynamics, while the second and the third ones are meant to relocate the torus and straighten the normal dynamics.

For the moment we solve the equations "modulo $\dot{\lambda}$ "; eventually $\delta\lambda$ will be uniquely chosen to kill the constant component of the given terms belonging to the kernel of $A - I$ and $[A, \cdot]$ respectively, and solve the cohomological equations.

In the following we will repeatedly apply Lemmata 1-2-3 and Cauchy's inequality. Furthermore, we do not keep track of constants - just note that they may

only depend on n and τ (from the Diophantine condition) and on $|G - \text{id}|_{s+\sigma}$ and $|P - ((\theta + 2\pi\alpha), A \cdot r)|_{s+\sigma}$, and refer to them as C .

– First, second equation has a solution

$$\dot{R}_0 = L_{1,A}^{-1}(\dot{Q}_0 + \dot{b} - \bar{b}),$$

where

$$\bar{b} = \int_{\mathbb{T}} \dot{Q}_0 + \dot{b} \frac{d\theta}{2\pi},$$

and

$$|\dot{R}_0|_s \leq \frac{C}{\gamma^2 \sigma^{\tau+n}} |\dot{Q}_0 + \dot{b}|_{s+\sigma}.$$

– Second, we have

$$\dot{\varphi}(\theta + 2\pi\alpha) - \dot{\varphi}(\theta) + p_1 \cdot \dot{R}_0 = \dot{q}_0 + \dot{\beta} - \bar{\beta},$$

where $\bar{\beta} = \int_{\mathbb{T}^n} \dot{q}_0 - p_1 \cdot R_0 + \dot{\beta} \frac{d\theta}{(2\pi)^n}$, hence

$$\dot{\varphi} = L_{\alpha}^{-1}(\dot{q}_0 + \dot{\beta} - \bar{\beta}),$$

satisfying

$$|\dot{\varphi}|_{s-\sigma} \leq \frac{C}{\gamma \sigma^{\tau+n+1}} |\dot{q}_0 + \dot{\beta}|_{s+\sigma}$$

– Third, the solution of equation in \dot{R}_1 is

$$\dot{R}_1 = L_{2,A}^{-1}(\tilde{Q}_1 + \dot{B} - \bar{B}),$$

having denoted $\tilde{Q}_1 = \dot{Q}_1 - (2P_2 \cdot \dot{R}_0 + \dot{R}_0'(\theta + 2\pi\alpha) \cdot p_1)$, and \bar{B} the average of $\tilde{Q}_1 + \dot{B}$. It satisfies

$$|\dot{R}_1|_{s-2\sigma} \leq \frac{C}{\gamma \sigma^{n+\tau}} |\tilde{Q}_1 + \dot{B}|_{s+\sigma}.$$

We now handle the unique choice of the correction $\delta\lambda = (\delta\beta, \delta b + \delta B \cdot r)$ given by $T_{\delta\lambda}$. If $\bar{\lambda} = (\bar{\beta}, \bar{b} + \bar{B} \cdot r)$, the map $f : \Lambda \rightarrow \Lambda$, $\delta\lambda \mapsto \bar{\lambda}$ is well defined. In particular when $G = \text{id}$, $\frac{\partial f}{\partial \delta\lambda} = -\text{id}$ and it will remain bounded away from 0 if G stays sufficiently close to the identity: let say $|G - \text{id}|_{s_0} \leq \varepsilon_0$, for $s_0 < s$. In particular, $-\bar{\lambda}$ is affine in $\delta\lambda$, the system to solve being triangular of the form $\int_{\mathbb{T}^n} a(G, \dot{Q}) + A(G) \cdot \delta\lambda = 0$, with diagonal close to 1 if the smallness condition above is assumed. Under these conditions f is a local diffeomorphism and there exists unique $\delta\lambda$ such that $f(\delta\lambda) = 0$, satisfying

$$|\delta\lambda| \leq \frac{C}{\sigma^{\tilde{\tau}}} |\delta Q|_{G,s+\sigma},$$

for some $\tilde{\tau} > 1$. We finally have

$$|\dot{G}|_{s-2\sigma} \leq \frac{C}{\gamma} \frac{1}{\sigma^{\tau''}} |\delta Q|_{G,s+\sigma}.$$

Now, from the definition of $\dot{G} = G'^{-1} \cdot \delta G$ we get $\delta G = G' \cdot \dot{G}$, hence similar estimates hold for δG :

$$|\delta G|_{s-\sigma} \leq \sigma^{-1} (1 + |G - \text{id}|_s) \frac{C}{\sigma^{\tau''''}} |\delta Q|_{G,s+\sigma}.$$

Eventually, equation (4.3) uniquely determines δP .

Up to redefining $\sigma' = \sigma/3$ and $s' = s + \sigma$, we have the wanted estimates for all $s', \sigma' : s' < s' + \sigma'$. \square

Proposition 4.2 (Boundness of ϕ''). *The bilinear map $\phi''(x)$*

$$\phi''(x) : (T_G \mathcal{G}_{s+\sigma}^\sigma \times \vec{U}_{s+\sigma}(\alpha, A) \times \Lambda)^{\otimes 2} \rightarrow \mathcal{A}(\mathbb{T}_s^n, \mathbb{T}_{\mathbb{C}}^n),$$

satisfies the estimates

$$|\phi''(x) \cdot \delta x^{\otimes 2}|_{G,s} \leq \frac{C''}{\sigma^{\tau''}} |\delta x|_{s+\sigma}^2,$$

C'' being a constant depending on $|x|_{s+\sigma}$.

Proof. Differentiating twice $\phi(x)$, yields

$$\begin{aligned} & -M\{[\delta G' \circ P \cdot \delta P + \delta G' \circ P \cdot \delta P + G'' \circ P \cdot \delta P^2 - (\delta G' \circ P + G'' \circ P \cdot \delta P) \cdot P' \cdot G'^{-1} \cdot \delta G \\ & - G' \circ P \cdot (\delta P' \cdot (-G'^{-1} \cdot \delta G' \cdot G'^{-1}) \cdot \delta G)] \circ G^{-1} + \\ & + [\delta G' \circ P \cdot \delta P + \delta G' \circ P \cdot \delta P + G'' \circ P \cdot \delta P^2 - (\delta G' \circ P + G'' \circ P \cdot \delta P) \cdot P' \cdot G'^{-1} \cdot \delta G \\ & - G' \circ P \cdot (\delta P' \cdot (-G'^{-1} \cdot \delta G' \cdot G'^{-1}) \cdot \delta G)]' \circ G^{-1} \cdot (-G'^{-1} \cdot \delta G) \circ G^{-1}\}. \end{aligned}$$

Once we precompose with G , the estimate follows. \square

Hypothesis of Theorem A.1 are satisfied, hence the existence of (G, P, λ) such that $Q = T_\lambda \circ G \circ P \circ G^{-1}$. Uniqueness and smoothness of the normal form follows from Propositions A.1 and A.2-A.3. Theorem 4.1 follows, hence Theorem A.

5. A GENERALIZATION OF RÜSSMANN'S THEOREM

Theorem A provides a normal form that does not rely on any non-degeneracy assumption; thus, the existence of a translated Diophantine, reducible torus will be subordinated to eliminating the "parameters in excess" (β, B) using a non-degeneracy hypothesis. We will implicitly solve $B = 0$ and $\beta = 0$ by using the normal frequencies as *free* parameters and a torsion hypothesis respectively. Rüssmann's classic result will be the immediate small dimensional case.

Elimination of B . Let $\Delta_m^s(\mathbb{R}) \subset \text{Mat}_m(\mathbb{R})$ be the open space of matrices with simple, real, non 0 eigenvalues. In $\mathbb{T}^n \times \mathbb{R}^m$, let us define

$$\widehat{U} = \bigcup_{I+A \in \Delta_m^s(\mathbb{R})} U(\alpha, I+A).$$

We recall that those $P's \in U(\alpha, I+A)$ are diffeomorphisms of the form

$$P(\theta, r) = (\theta + 2\pi\alpha + O(r), (I+A) \cdot r + O(r^2)),$$

on a neighborhood of $\mathbb{T}^n \times \{0\}$.

The following theorem is an intermediate, yet fundamental result to prove the translated torus Theorem C and holds without requiring any torsion assumption on the class of diffeomorphisms.

Theorem 5.1 (Twisted Torus of co-dimension 1). *For every $P^0 \in U_{s+\sigma}(\alpha, I + A^0)$ with α Diophantine, and $I + A^0 \in \Delta_m^s(\mathbb{R})$, there is a germ of C^∞ -maps*

$$\psi : V_{s+\sigma} \rightarrow \mathcal{G}_s \times \widehat{U}_s \times \Lambda(\beta, b), \quad Q \mapsto (G, P, \lambda),$$

at $P^0 \mapsto (\text{id}, P^0, 0)$, such that $Q = T_\lambda \circ G \circ P \circ G^{-1}$, where $\lambda = (\beta, b) \in \mathbb{R}^{n+1}$.

Corollary 5.1 (Twisted torus). *If 1 does not belong to the spectrum of $I + A^0$, the translation correction $b = 0$.*

Proof. Denote ϕ_A the operator ϕ , as now we want A to vary. Let us identify with \mathbb{R}^m the space of diagonal matrices in $\text{Mat}_n(\mathbb{R})$ and define the map

$$\hat{\psi} : \mathbb{R}^m \times V_{s+\sigma} \rightarrow \mathcal{G}_s \times U_s(\alpha, I + A) \times \Lambda, \quad (A, Q) \mapsto \hat{\psi}_A(Q) := \phi_A^{-1}(Q) = (G, P, \lambda)$$

in the neighborhood of (A^0, P^0) , such that $Q = T_\lambda \circ G \circ P \circ G^{-1}$.

By writing P^0 as

$$P^0(\theta, r) = (\theta + 2\pi\alpha + O(r), (I + A_0 - \delta A) \cdot r + \delta A \cdot r + O(r^2)),$$

we remark that $P^0 = T_\lambda \circ P_A$, where

$$\lambda = (0, B(A) = (A^0 - A) \cdot (I + A)^{-1}),$$

and $P_A = (\theta + 2\pi\alpha + O(r), (I + A) \cdot r + O(r^2))$, $A = A^0 - \delta A$.²

According to Theorem A, $\phi_A(\text{id}, P_A, \lambda) = P^0$, thus locally for all A close to A^0 we have

$$\hat{\psi}(A, P^0) = (\text{id}, P_A, B \cdot r), \quad B(A, P^0) = (A^0 - A) \cdot (I + A)^{-1} = \delta A \cdot (I + A^0 - \delta A)^{-1}$$

and, in particular $B(A^0, P^0) = 0$ and

$$\frac{\partial B}{\partial A}|_{A=A^0} = -(I + A^0)^{-1},$$

which is invertible, due to hypothesis on the spectrum of A^0 . Hence $A \mapsto B(A)$ is a local diffeomorphism and by the implicit function theorem (in finite dimension) locally for all Q close to P^0 there exists a unique \bar{A} such that $B(\bar{A}, Q) = 0$. It remains to define $\psi(Q) = \hat{\psi}(\bar{A}, Q)$. \square

The proof of Corollary 5.1 is immediate, by conditions (1.4).

Remark 5.1. This twisted-torus theorem relies on the peculiarity of the normal dynamics of the torus T_0^n . The direct applicability of the implicit function theorem is subordinated to the fact that no arithmetic condition is required on the characteristic (normal) frequencies; beyond that, since having simple, real eigenvalues is an open property, the needed correction B is indeed a diagonal matrix, so that the number of free frequencies (parameters) is enough to solve, implicitly, $B(A) = 0$. The generic case of complex eigenvalues is more delicate since one should guarantee that corrections $A^0 + \delta A$ at each step, satisfy Diophantine condition (1.2). It seems reasonable to think that one would need more parameters to control this issue, and verify that the measure of such stay positive; see [11].

²The terms $O(r)$ and $O(r^2)$ contain a factor $(1 - \delta A \cdot (1 + A_0)^{-1})$.

Elimination of β . If Q satisfy a torsion hypothesis, the existence of a translated Diophantine torus can be proved.

Theorem 5.2 (Translated Diophantine torus). *Let α be Diophantine. On a neighborhood of $\mathbb{T}^n \times \{0\} \subset \mathbb{T}^n \times \mathbb{R}^n$, let $P^0 \in U(\alpha, I + A^0)$ be a diffeomorphism of the form*

$$P^0(\theta, r) = (\theta + 2\pi\alpha + p_1(\theta) \cdot r + O(r^2), (I + A^0) \cdot r + O(r^2)),$$

with $I + A^0$ of simple, real non 0 eigenvalues and such that

$$\det \left(\int_{\mathbb{T}^n} p_1(\theta) d\theta \right) \neq 0.$$

If Q is close enough to P^0 there exists a unique A , close to A^0 , and a unique $(G, P, b) \in \mathcal{G} \times U(\alpha, I + A) \times \mathbb{R}^n$ such that $Q = T_b \circ G \circ P \circ G^{-1}$.

Phrasing the thesis, the graph of $\gamma = R_0 \circ \varphi^{-1}$ is a translated torus on which the dynamics is conjugated to $R_{2\pi\alpha}$ by φ (remember the form of $G \in \mathcal{G}$ given in (1.3)). Before proceeding with the proof of Theorem 5.2, let us consider a parameter $c \in B_1^n(0)$ (the unit ball in \mathbb{R}^n) and the family of maps defined by $Q_c(\theta, r) := Q(\theta, c + r)$ obtained by translating the action coordinates. Considering the corresponding normal form operators ϕ_c , the parametrized version of Theorem A follows readily.

Now, if Q_c is close enough to P_c^0 , Theorem 5.1 asserts the existence of $(G_c, P_c, \lambda_c) \in \mathcal{G} \times U(\alpha, A) \times \Lambda(\beta, b)$ such that

$$Q_c = T_\lambda \circ G_c \circ P_c \circ G_c^{-1}.$$

Hence we have a family of tori parametrized by $\tilde{c} = c + \int_{\mathbb{T}^n} \gamma \frac{d\theta}{(2\pi)^n}$,

$$Q(\theta, \tilde{c} + \tilde{\gamma}(\theta)) = (\beta(c) + \varphi \circ R_{2\pi\alpha} \circ \varphi^{-1}(\theta), b(c) + \tilde{c} + \tilde{\gamma}(\varphi \circ R_{2\pi\alpha} \circ \varphi^{-1}(\theta))),$$

where $\gamma := R_0 \circ \varphi^{-1}$ and $\tilde{\gamma} = \gamma - \int_{\mathbb{T}} \gamma \frac{d\theta}{2\pi}$.

Proof. Let $\hat{\varphi}$ be the function defined on \mathbb{T}^n taking values in $\text{Mat}_n(\mathbb{R})$ that solves the (matrix of) difference equation

$$\hat{\varphi}(\theta + 2\pi\alpha) - \hat{\varphi}(\theta) + p_1(\theta) = \int_{\mathbb{T}^n} p_1(\theta) \frac{d\theta}{(2\pi)^n},$$

and let $F : (\theta, r) \mapsto (\theta + \hat{\varphi}(\theta) \cdot r, r)$. The diffeomorphism F restricts to the identity at \mathbb{T}_0^n . At the expense of substituting P^0 and Q with $F \circ P^0 \circ F^{-1}$ and $F \circ Q \circ F^{-1}$ respectively, we can assume that

$$P^0(\theta, r) = (\theta + 2\pi\alpha + p_1 \cdot r + O(r^2), (I + A^0) \cdot r + O(r^2)), \quad p_1 = \int_{\mathbb{T}^n} p_1(\theta) \frac{d\theta}{(2\pi)^n}.$$

The germs obtained from the initial P^0 and Q are close to one another.

The proof will follow from Theorem 5.1 and the elimination of the parameter $\beta \in \mathbb{R}^n$ obstructing the rotation conjugacy.

In line with the previous reasoning, we want to show that the map $c \mapsto \beta(c)$ is a local diffeomorphism. It suffices to show this for the trivial perturbation P_c^0 . The Taylor expansion of P_c^0 directly gives the normal form. In particular $b(c) = (I + A^0) \cdot c + O(c^2)$, while the map $c \mapsto \beta(c) = p_1 \cdot c + O(c^2)$ is such that $\beta(0) = 0$

and $\beta'(0) = p_1$ which is invertible by twist hypothesis, thus a local diffeomorphism. Hence, the analogous map for Q_c , which is a small C^1 -perturbation, is a local diffeomorphism too and, together with Theorem 5.1, there exists a unique $c \in \mathbb{R}^n$ and $A \in \text{Mat}_n(\mathbb{R})$, such that $(\beta, B) = (0, 0)$. \square

Remark 5.2. The theorem holds also on $\mathbb{T}^n \times \mathbb{R}^m$, with $m \geq n$, requiring that

$$\text{rank} \left(\int_{\mathbb{T}^n} p_1(\theta) d\theta \right) = n.$$

This guarantees that $c \mapsto \beta(c)$ is submersive, but c solving $\beta(c) = 0$ would no more be uniquely determined.

Remark 5.3. Theorem 5.2 generalizes the classic translated curve theorem of Rüssmann in higher dimension, in the case of normally hyperbolic systems such that A has simple, real, non 0 eigenvalues, for general perturbations.

We stress the fact that if P^0 was of the form

$$P^0(\theta, r) = (\theta + 2\pi\alpha + O(r), I \cdot r + O(r^2)),$$

like in the original frame studied by Rüssmann, we would need a whole matrix $B \in \text{Mat}_n(\mathbb{R})$ in order to solve the homological equations, and, having just n characteristic frequencies at our disposal, it is hopeless to completely solve $B = 0$ and eliminate the whole obstruction. The torus would not be just translated.

APPENDIX A. INVERSE FUNCTION THEOREM & REGULARITY OF ϕ

We state here the implicit function theorem we use to prove Theorem A as well as the regularity statements needed to guarantee uniqueness and smoothness of the normal form. These results follow from Féjóz [12, 13]. Remark that we endowed functional spaces with weighted norms and bounds appearing in propositions 4.1-4.2 may depend on $|x|_s$ (as opposed to the analogue statements in [12, 13]); for the corresponding proofs taking account of these (slight) differences we send the reader to [18, 19] and the proof of Moser's theorem therein.

Let $E = (E_s)_{0 < s < 1}$ and $F = (F_s)_{0 < s < 1}$ be two decreasing families of Banach spaces with increasing norms $|\cdot|_s$ and let $B_s^E(\sigma) = \{x \in E : |x|_s < \sigma\}$ be the ball of radius σ centered at 0 in E_s .

On account of composition operators, we additionally endow F with some deformed norms which depend on $x \in B_s^E(s)$ such that

$$|y|_{0,s} = |y|_s \quad \text{and} \quad |y|_{\hat{x},s} \leq |y|_{x,s+|x-\hat{x}|_s}.$$

Consider then operators commuting with inclusions $\phi : B_{s+\sigma}^E(\sigma) \rightarrow F_s$, with $0 < s < s + \sigma < 1$, such that $\phi(0) = 0$.

We then suppose that if $x \in B_{s+\sigma}^E(\sigma)$ then $\phi'(x) : E_{s+\sigma} \rightarrow F_s$ has a right inverse $\phi'^{-1}(x) : F_{s+\sigma} \rightarrow E_s$ (for the particular operators ϕ of this work, ϕ' is both left and right invertible).

ϕ is supposed to be at least twice differentiable.

Let $\tau := \tau' + \tau''$ and $C := C'C''$.

Theorem A.1 (Inverse function theorem). *Further assume*

$$(A.1) \quad |\phi'^{-1}(x) \cdot \delta y|_s \leq \frac{C'}{\sigma^{\tau'}} |\delta y|_{x, s+\sigma}$$

$$(A.2) \quad |\phi''(x) \cdot \delta x^{\otimes 2}|_{x, s} \leq \frac{C''}{\sigma^{\tau''}} |\delta x|_{s+\sigma}^2, \quad \forall s, \sigma : 0 < s < s + \sigma < 1$$

C' and C'' depending on $|x|_{s+\sigma}$, $\tau', \tau'' \geq 1$.

For any s, σ, η with $\eta < s$ and $\varepsilon \leq \eta \frac{\sigma^{2\tau}}{2s\tau C^2}$ ($C \geq 1, \sigma < 3C$), ϕ has a right inverse $\psi : B_{s+\sigma}^F(\varepsilon) \rightarrow B_s^E(\eta)$. In other words, ϕ is locally surjective:

$$B_{s+\sigma}^F(\varepsilon) \subset \phi(B_s^E(\eta)).$$

Proposition A.1 (Lipschitz continuity of ψ). *Let $\sigma < s$. If $y, \hat{y} \in B_{s+\sigma}^F(\varepsilon)$ with $\varepsilon = 3^{-4\tau} 2^{-16\tau} \frac{\sigma^{6\tau}}{4C^3}$, the following inequality holds*

$$|\psi(y) - \psi(\hat{y})|_s \leq L |y - \hat{y}|_{x, s+\sigma},$$

with $L = 2C'/\sigma^{\tau'}$. In particular, ψ being the unique local right inverse of ϕ , it is also its unique left inverse.

Proposition A.2 (Smooth differentiation of ψ). *Let $\sigma < s < s + \sigma$ and ε as in proposition A.1. There exists a constant K such that for every $y, \hat{y} \in B_{s+\sigma}^F(\varepsilon)$ we have*

$$|\psi(\hat{y}) - \psi(y) - \phi'^{-1}(\psi(y))(\hat{y} - y)|_s \leq K(\sigma) |\hat{y} - y|_{x, s+\sigma}^2,$$

and the map $\psi' : B_{s+\sigma}^F(\varepsilon) \rightarrow L(F_{s+\sigma}, E_s)$ defined locally by $\psi'(y) = \phi'^{-1}(\psi(y))$ is continuous. In particular ψ has the same degree of smoothness of ϕ .

It is sometimes convenient to extend ψ to non-Diophantine characteristic frequencies (α, A) . Whitney smoothness guarantees that such an extension exists. Let suppose that $\phi(x) = \phi_\nu(x)$ depends on some parameter $\nu \in B^k$ (the unit ball of \mathbb{R}^k) and that it is C^1 with respect to ν and that estimates on ϕ'_ν and ϕ''_ν are uniform with respect to ν over some closed subset D of \mathbb{R}^k .

Proposition A.3 (Whitney differentiability). *Let us fix ε, σ, s as in proposition A.1. The map $\psi : D \times B_{s+\sigma}^F(\varepsilon) \rightarrow B_s^E(\eta)$ is C^1 -Whitney differentiable and extends to a map $\psi : \mathbb{R}^{2n} \times B_{s+\sigma}^F(\varepsilon) \rightarrow B_s^E(\eta)$ of class C^1 . If ϕ is C^k , $1 \leq k \leq \infty$, with respect to ν , this extension is C^k .*

APPENDIX B. INVERSION OF A HOLOMORPHISM OF \mathbb{T}_s^n

We present here a classical result and a lemma that justify definition of the normal form operator ϕ defined in section 2.3.

Complex extensions of manifolds are defined at the help of the ℓ^∞ -norm.

Let

$$\mathbb{T}_{\mathbb{C}}^n = \mathbb{C}^n / 2\pi\mathbb{Z}^n \quad \text{and} \quad \mathbb{T}_s^n = \mathbb{T}_{\mathbb{C}}^n \times \mathbb{C}^m,$$

$$\mathbb{T}_s^n = \left\{ \theta \in \mathbb{T}_{\mathbb{C}}^n : |\theta| := \max_{1 \leq j \leq n} |\operatorname{Im} \theta_j| \leq s \right\}, \quad \mathbb{T}_s^n = \{(\theta, r) \in \mathbb{T}_{\mathbb{C}}^n : |(\operatorname{Im} \theta, r)| \leq s\},$$

where $|\operatorname{Im} \theta, r| := \max_{1 \leq j \leq n} \max(|\operatorname{Im} \theta_j|, |r_j|)$.

Let also define $\mathbb{R}_s^n := \mathbb{R}^n \times (-s, s)$ and consider the universal covering of \mathbb{T}_s^n , $p : \mathbb{R}_s^n \rightarrow \mathbb{T}_s^n$.

Theorem B.1. *Let $v : \mathbb{T}_s^n \rightarrow \mathbb{C}^n$ be a vector field such that $|v|_s < \sigma/n$. The map $\operatorname{id} + v : \mathbb{T}_{s-\sigma}^n \rightarrow \mathbb{R}_s^n$ induces a map $\varphi = \operatorname{id} + v : \mathbb{T}_{s-\sigma}^n \rightarrow \mathbb{T}_s^n$ which is a biholomorphism and there is a unique biholomorphism $\psi : \mathbb{T}_{s-2\sigma}^n \rightarrow \mathbb{T}_{s-\sigma}^n$ such that $\varphi \circ \psi = \operatorname{id}_{\mathbb{T}_{s-2\sigma}^n}$.*

In particular the following hold:

$$|\psi - \operatorname{id}|_{s-2\sigma} \leq |v|_{s-\sigma}$$

and, if $|v|_s < \sigma/2n$

$$|\psi' - \operatorname{id}|_{s-2\sigma} \leq \frac{2}{\sigma} |v|_s.$$

For the proof we send again to [18, 19].

Corollary B.1 (Well definition of the normal form operator ϕ). *For all s, σ if $G \in \mathcal{G}_{s+\sigma}^{\sigma/n}$, then $G^{-1} \in \mathcal{A}(\mathbb{T}_s^n, \mathbb{T}_{s+\sigma}^n)$.*

Proof. We recall the form of $G \in \mathcal{G}_{s+\sigma}^{\sigma/n}$:

$$G(\theta, r) = (\varphi(\theta), R_0(\theta) + R_1(\theta) \cdot r).$$

G^{-1} reads

$$G^{-1}(\theta, r) = (\varphi^{-1}(\theta), R_1^{-1} \circ \varphi^{-1}(\theta) \cdot (r - R_0 \circ \varphi(\theta))).$$

Up to rescaling norms by a factor $1/2$ like $\|x\|_s := \frac{1}{2}|x|$, the statement is straightforward and follows from theorem B.1. By abuse of notations, we keep on indicating $\|x\|_s$ with $|x|_s$. \square

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